

# Current Trends in Mathematical Cosmology

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February 7, 2008

## Abstract

We present an elementary account of mathematical cosmology through a series of important unsolved problems. We introduce the fundamental notion of *a cosmology* and focus on the issue of singularities as a theme unifying many current, seemingly unrelated trends of this subject. We discuss problems associated with the definition and asymptotic structure of the notion of cosmological solution and also problems related to the qualification of approximations and to the ranges of validity of given cosmologies.

## 1 Introduction

*Circumstantial evidence is a very tricky thing. It may seem to point very straight to one thing, but if you shift your own point of view a little, you may find it pointing in an equally uncompromising manner to something entirely different. (Sherlock Holmes)*

In any field of applied mathematics one starts by carefully identifying the basic object of study, one that contains the essential parameters of the problem, to be determined

by later analysis. The mathematical methods of cosmology which promise to be useful, even essential to the nature of problems one typically encounters come from two different sources, namely, differential geometry and dynamical systems theory. Indeed, *mathematical cosmology*, a largely unexplored but highly interesting and promising area of research, may be loosely defined as that separate discipline in applied mathematics which lies in the differentiable world of geometry and dynamical systems borrowing heavily from both areas and contributing back constantly new problems and ideas not only to both of these mathematical fields but also to the closely related physical and observational cosmology.

The basic object of study in any mathematical approach to cosmological problems is that of a *cosmological model*. Let us explain briefly what a cosmological model is and how we can generate interesting models. We call a *cosmology* the result of combining the mathematical theorizing that goes into the construction of a cosmological model with the observational data that are available in the astronomical literature. In the following, however, we shall ignore this difference between a cosmology and a (cosmological) model and use both words indistinguishably to describe this fundamental notion of cosmological modelling.

In this paper we lay the foundations of mathematical cosmology in a manner suitable for the nonspecialist, focusing on the fundamental mathematical problems which single out this field as a separate component within applied mathematics and mathematical physics. The presentation is elementary and is addressed to those who need a general overview before plugging in the excruciating details.

In the next Section, we introduce the idea of a cosmology as a basic unknown of this subject. Section 3 discusses the notion of cosmological law and shows how the singularity problem, a central issue in this field, is used to orient the whole of mathematical cosmology research around three basic themes, namely, global evolution, approximations and range of validity of a cosmology. Sections 4 to 6 describe in more detail these basic avenues of research expanding on several open questions relevant to each theme. We conclude in Section 7 with some more general comments on the nature of mathematical modelling in cosmology.

This is meant to be a short review of a huge subject and therefore we apologize in advance for many superficial passages, or possible omissions of important ideas and

works by fellow mathematical cosmologists over the past decades. In this sense the bibliography contains some works which the author has found relevant in the preparation of this paper and is only meant to be a useful guide to those interested in pursuing this beautiful subject further. It contains mainly review articles and books and is not to be regarded as a declaration of the most important sources in our field.

## 2 Cosmologies

*There is nothing so unnatural as the commonplace. (Sherlock Holmes)*

There are three essential elements that go into a cosmology:

- A cosmological spacetime (CS)
- A theory of gravity (TG)
- A collection of matterfields (MF)

A cosmology is a particular way of combining these three basic elements into a meaningful whole:

$$\text{Cosmology} = \text{CS} + \text{TG} + \text{MF}. \quad (1)$$

There is a basic hierarchy of CSs according to the degree of exact symmetry present. We basically start with a smooth manifold  $\mathcal{M}$  and impose a Lorentzian metric  $g_{ab}$  on  $\mathcal{M}$  which admits a number of symmetries. Generally speaking, the CS's hierarchy list is:

1. Isotropic (Friedmann-Robertson-Walker) spacetimes
2. Homogeneous (Bianchi) spacetimes
3. Inhomogeneous spacetimes
4. Generic spacetimes

This list is one of decreasing symmetry, and so increasing generality, as we move from top to bottom and comprises four families of CSs. The last family, generic spacetimes,

has no symmetry whereas the isotropic spaces correspond to the simplest (and perhaps unphysical), highest-symmetry toy models that exist.

TG too, fortunately or not depending on how one looks at it, come in great variety. A partial list of important families of theories which include gravity-a necessary ingredient for the modern construction of cosmologies-is:

1. General relativity (GR)
2. Higher derivative gravity theories (HDG)
3. Scalar-tensor theories (ST)
4. Superstring theories (SS)

It is widely accepted today, after the pioneering work of Hawking, Geroch and Penrose in the late sixties (see [1] for an account of these results) that GR leads to singularities in the early development of generic CSs and consequently one needs a better TG to account for early cosmological events in a consistent way. The above list is motivated partially by these results and comprises, besides GR, modifications involving higher derivatives, scalarfield-curvature couplings as well as supersymmetric ideas in the formulation of entries 2, 3 and 4 above respectively.

Matterfields also come in an ambitious shopping list of interesting candidates which may have played an important role during different epochs in the history of the universe. For example, we can consider:

1. Vacuum
2. Fluids
3. Scalar fields
4.  $n$ -form fields

See Table 1 for a summary. Definition (1) above is a very broad one. The simplest and best studied (relativistic) cosmology of physical interest is the (FRW/GR/Fluid) cosmology. This is in fact the cosmology discussed in many textbooks on the subject under the heading ‘Relativistic Cosmology’, but we may obviously attempt to construct and analyze other possible cosmologies. For example, we can consider the families:

Table 1: Three essential elements comprising a cosmology according to Eq. (1). Also shown are several members of each particular element.

<i>Cosmologies</i>		
Theories of gravity	Cosmological spacetimes	Matterfields
General Relativity	Isotropic	Vacuum
Higher Derivative Gravity	Homogeneous	Fluids
Scalar-Tensor theories	Inhomogeneous	Scalarfields
String theories	Generic	$n$ -form fields

- FRW/GR/vacuum
- Bianchi/ST/fluid
- Inhomogeneous/String/ $n$ -form
- Generic/GR/vacuum

and so on. For visualization purposes, we can consider the 3-dimensional 'space' of all cosmologies with coordinates (**CS**,**TG**,**MF**) the 'points' of which represent different cosmologies<sup>1</sup>. For example, the category ( $\cdot$ /GR/ $\cdot$ ), a higher dimensional subspace in the basic cosmology space, is the best studied cosmology so far – see the excellent review [3].

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<sup>1</sup>We are well aware of the danger present here of being too pedantic or intimidating for most readers when discussing such 'meta-cosmological' issues. However, if one wishes to ponder for a minute about the difficulties involved in the construction of such a space, we note that, to the best of our knowledge, it is unclear at present how to put an ordering in the three axes **CS**, **TG** and **MF**. The older notion of superspace (cf. [2]) provides such an order relation only for the **CS**-axis of our cosmology-space. The construction of analogous orderings for lagrangians corresponding to **TG** and **MF** leading to a consistent topology on the space of all cosmologies is beyond the scope of the present article and could constitute an interesting avenue of research.

### 3 Global evolution, approximations and range of validity

*Functions, just like living beings, are characterized by their singularities. (P. Montel)*

The tool we use to translate the above into a consistent mathematical language is the Action Principle. We use this tool to formulate precisely the notions of a **TG** and that of a **MF**. So, how do we construct a cosmology? Pick up a spacetime from the cosmological hierarchy list, choose a gravity theory and one or more matterfields, tie them together through the Action Principle and try to explain the observed facts in terms of the consequences of the application of the variational principle (for a detailed mathematical introduction see [4]).

Through the action principle, the resulting *cosmological equations* one obtains by starting from a *cosmological lagrangian* and using symmetry or other phenomenological considerations contain the basic properties, to be unraveled, of any cosmology. There are many questions that can be asked for any such set of equations, leading in this way to many different fundamental trends in theoretical cosmology today and of course to the rest of this paper. Before we proceed to discuss some of these problems, however, we pause to explain what a cosmological equation has to do with another basic notion that we shall encounter, *the cosmological law*.

We distinguish, for the purpose of orientation, two kinds of such laws, that is, *fundamental* and *effective* cosmological laws<sup>2</sup>. We talk of a fundamental cosmological law when we are faced with a set of equations of the form (**TG**/**MF**) that is, when we have not imposed any symmetry in the underline spacetime. Now it is probably somewhat surprising or even misleading to call, say, the full Einstein equations,  $G_{ab} = kT_{ab}$ , a *cosmological law* for, any such set of equations contains much more than cosmological solutions eg., it contains black holes or gravitational waves. The only justification for this terminology

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<sup>2</sup>There is another kind of cosmological ‘law’, the set of ideas that goes by the name of *The Anthropic Principle* [5]. However, the usage of the word ‘law’ we adopt here only includes those that are formulated in the form of dynamical systems. It is, we believe, an intriguing question whether the Anthropic Principle has some hidden dynamical meaning and, if yes, how could this be possibly framed in the form of a differential equation.

is that in the full, ‘unconstrained’, case one is interested in the behaviour of the *whole* spacetime and this does not have *a priori* imposed on it any specific assumption that would lead to other kinds of solutions (for example asymptotic flatness etc). Hence, a fundamental cosmological law has only mild assumptions on  $(\mathcal{M}, g_{ab})$ , for instance it can be taken to be globally hyperbolic with only some physically reasonable energy condition imposed on the matter content.

On the other hand any imposition of symmetry or other reduction principle on  $(\mathcal{M}, g_{ab})$  and the matter fields leads to *effective cosmological laws*, that is, to more special sets of differential equations which are thus obtained as byproducts of a given fundamental cosmological law. Thus for instance, imposing on the fundamental law (GR/matter) the usual isotropic conditions and requiring the admission of a perfect fluid matter source we obtain the effective law (FRW/GR/Perfect Fluid) – a set of equations for the time evolution of the scale factor and the fluid parameters.

Even at this stage we easily realize the importance of genericity vs. symmetry or fundamental vs. effective cosmological laws. This dichotomy raises a basic question of principle: Taking for granted the extreme difficulty to handle mathematically any fundamental law to produce strong results, how could we ever be sure that we obtained reliable or generic results while working at the ‘lower’ levels of effective laws? Soon we shall be more specific and have more to say.

We now return to our basic theme. The problem of possible break-down or singularity in the future or past of any given cosmology is the most basic problem in mathematical cosmology and affects all cosmologies. (Almost) all cosmological solutions are likely to form singularities in a finite time. This forces us to consider the following two fundamental questions which frame the singularity problem in cosmology: ‘What do we mean by a cosmological solution?’ and, ‘What is the range of validity of a given cosmology?’. The singularity problem has several interrelated offshoots:

- Where are the cosmological singularities to be found?
- Why do cosmological solutions have the tendency to develop singularities?
- What is the nature of the cosmological singularity?
- Can we continue the solution past the singularity?

Indeed the singularity problem can be efficiently used to signpost the current status of our subject. What is then the present state of mathematical cosmology? Overall our present efforts are directed to

1. **Global evolution:** *Understand the global evolution of solutions to the cosmological equations resulting from all possible available laws*
2. **Qualifications of approximations:** *Evaluate the various approximations (especially matterfields) involved*
3. **Range of validity:** *Decide on the range of validity of the various cosmologies and clarify the meaning of the notion of cosmological solution*

In the next three Sections we take up in turn each one of the above fundamental trends and present in some detail some of the basic sub-topics that the community of mathematical cosmologists has found interesting and occupied itself with over the years.

## 4 Asymptotic cosmological states

*If there is no time, there is no space. (Yvonne Choquet-Bruhat)*

The issue of determining the global cosmological evolution, *alias* the problem of the asymptotic cosmological states, is a very basic and by and large open research problem in mathematical cosmology today. In view of the impossibility of meaningfully framing universal boundary conditions in cosmology this problem becomes particularly important in any attempt to understand the long term behaviour of cosmological systems. Its two components, dynamics in the positive, expanding direction and that in the negative or contracting direction present us with different issues, not least because of their different physical interpretation.

The dimensionality of cosmological dynamical systems varies from one (in the case of the simplest (FRW/GR) cosmologies) to infinity (generic cosmologies) and most of them are typically formulated in higher than two dimensions. This, together with the essential nonlinearities present in any cosmological law from which these systems are derived, results in making an already difficult subject even more demanding (and interesting!).



Attempting to reconstruct and classify the known results in the dynamics of cosmology according to dimension, gathering all systems of equal dimensionality together and discovering common features (‘what do we know about 1D cosmologies, 2D cosmologies, etc?’) might be an interesting project.

A general feature of the dynamics of cosmologies in the contracting direction is that things typically tend to become more complicated. The well-known BKL approximation scheme for approaching the singularity is the typical example [6, 7]. Qualifying the dynamics of contracting cosmologies has been a central problem in mathematical cosmology for many years. The pioneering work of Barrow in the early eighties on (Bianchi/GR) contracting cosmologies [8] established the connection between the complicated patterns of oscillations present in BKL and in the hamiltonian picture of Misner [9] and the theory of chaotic dynamical systems, thus opening up a whole new chapter in mathematical cosmology. After these works there has been a large body of literature connected with the issue of chaotic behaviour in different cosmologies continuing even to this day with many open problems still remaining (see also the following Section).

The behaviour of cosmologies in the expanding direction, however, appears to be of a completely different nature at least as far as the types of questions with which one is concerned. Chaotic behaviour in the future does not appear to be a typical feature of an expanding cosmology. Instead one is content in asking questions having to do with *stability*, *attractors* and *bifurcations*.

Typical examples for the stability and asymptotic stability of a given set of solutions within a cosmology include the stability problem of isotropic or homogeneous solutions with respect to perturbations either in the given theory of gravity or in a larger set of gravity theories (see, for example, [10], [11], [12]).

The attractor properties of exact solutions of physical interest, eg., inflationary, have occupied a great number of papers in the literature. Many of these results describe analytic ways of how a given solution approaches, or is approached by, another set of solutions. However, a rigorous general definition via dynamical systems theory of the notion of cosmological attractor that will prove useful in specific applications is still lacking. (The recent book [13] reviews some of these problems in the modern language of dynamical systems and contains basic results about equilibrium points, limit sets etc for the (Bianchi/GR) family.)

Many cosmologies are formulated as a set of dynamical equations with parameters for example, in the (Bianchi/GR) family there are three such parameters describing the passage from one Bianchi type to the next. Hence, the dynamics of cosmologies in many cases present us with interesting *bifurcation problems*. To our knowledge bifurcation theory (cf. for instance, [14]) has not been considered in mathematical cosmology up to now in any systematic way.

## 5 Qualifying the approximations

*Most people, if you describe a train of events to them, will tell you what the result would be. They can put those events together in their minds, and argue from them that something will come to pass. There are few people, however, who, if you told them a result, would be able to evolve from their own inner consciousness what the steps were which led up to that result. This power is what I mean when I talk of reasoning backward, or analytically. (Sherlock Holmes)*

In the direction of qualifying the approximations used a basic problem is to make sense of the aforementioned BKL oscillatory pattern of approach towards the initial cosmological singularity. We know that this is a local and piecewise scheme for describing the general approach to the singularity in, at least, the (BianchiIX/GR) category. It is well-known that this oscillatory pattern of approach to the singularity is disrupted by the inclusion of a scalar field [15, 16], in which case a monotonic approach to the singularity occurs, but the BKL phenomenon returns in the additional presence of a vector field as it was first noted in [15]. A rigorous analysis of this basic behaviour of (Bianchi/GR) cosmologies near the spacetime singularity was first given by Bogoyavlenski and Novikov in [17] using the method of maximal conformal compactification<sup>3</sup>.

Aside from the notoriously difficult problem of deciding *how* local this behaviour is, we know that these same patterns occur in many other cosmologies (taking into account conformal dualities between different cosmologies which typically transform them into

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<sup>3</sup>This analysis, among many other important results, is described in detail in the advanced book by Bogoyavlenski, [18].

the ‘kernel’ category ( $\cdot$ /GR/scalarfields). This applies, for instance, to the (BianchiIX /HDG) category [19] and certain members of the (BianchiIX /ST) family. The existence of a BKL scheme in higher dimensional (BianchiIX /GR) cosmologies is known [20] to be sensitive to the number of spacetime dimensions. Recently, the BKL problem in the context of certain (BianchiIX/String) cosmologies was considered in [21, 22] and essentially similar behaviours appear. This is due to the effect of scalar or vector fields on the vacuum behaviour of these cosmologies as well as to the number of dimensions.

However, the question of the genericity of the aforementioned behaviour persists. Is the general solution of a generic vacuum cosmology locally a Mixmaster (oscillatory) one? Recent work [25] shows that the answer to this question is indeed ‘yes’ in the (Inhomogeneous/GR) category.

A second, equally important, aspect of the issue of qualifying the approximations relates to the observational cosmology program developed by Ellis and his collaborators over the years [26, 27]. Without the assumption that we do not occupy a privileged position, the isotropy of the cosmic microwave background does not imply an isotropic spacetime geometry. Thus one proceeds by adding a new structure (a **CS**, a **TG** etc), one at a time, and examines what can be inferred about the geometry of the universe directly from observations with minimal assumptions.

This program has been implemented within GR and it would be very interesting to see what happens if a similar approach is taken up in other cosmologies based on extensions of general relativity such as, HDG or ST. For example, it can be shown [26] that without the use of a **TG** (in this case GR) one cannot obtain a ‘distance-redshift’ relation nor can prove, on the basis of observations alone, that our spacetime is spherically symmetric around us nor determine the sign of the curvature of the spatial sections. It is unknown whether these results are valid in the context of a HDG or a ST cosmology.

Hence a general ‘fitting’ problem may be formulated for cosmologies. Given a ‘lumpy’ cosmology and another ‘ideal’ one, the question arises as to how to determine a best-fit between the two. This question has been investigated in [27] within the context of general relativity. It is unclear whether similar results are valid in other cosmologies.

## 6 Ranges of validity

*The principal difficulty in your case lay in the fact of there being too much evidence. What was vital was overlaid and hidden by what was irrelevant. Of all the facts which were presented to us we had to pick just those which we deemed to be essential, and then piece them together in their order, so as to reconstruct this very remarkable chain of events. (Sherlock Holmes)*

The last avenue of research, the range of validity of a given cosmology, has at least two offshoots. The first can be well described by a remark of D. Christodoulou [28] (although made in the asymptotically flat context, assuming that the strong censorship conjecture turns out to be false) referring to the question of whether or not a given system which develops singularities at some finite time during its classical evolution necessarily requires the exit from the classical phase and a subsequent entrance to a quantum regime in order to have a meaningful description of the evolution past the singularity:

... for, it is argued, that a physical system which initially lies within the realm of validity of the theory would evolve into a system which lies outside this realm and we would be compelled to enter the domain of a quantum theory to obtain a valid description. A breakdown of this type does indeed occur in the Newtonian theory during stellar collapse to a neutron star or a black hole. However, this is by no means the only possibility. Another alternative is what very likely occurs in compressible fluid flow past the starting point of shock formation. We then have a new concept of solution for which complete regularity does not hold but singularities are of a milder character, the shocks. The subsequent evolution of these may be fully characterized by the classical theory, even though their microscopic structure, which resolves the discontinuities, is accessible only to a molecular description ...

Perhaps the generic singularities in cosmology, as predicted by the Hawking singularity theorems [1], are similar to shocks for many cosmologies. Do we then need a theory of quantum gravity in order to describe the cosmological evolution after or near cosmological singularities? What are the true ranges of validity of our available cosmologies?

Unravelling the *nature* of cosmological singularities is central to a beginning of understanding possible answers to this question.

This brings up a new dimension to the range of validity issue one connected with the view that there is no single cosmology which describes the universe at all times but some cosmologies may be better adapted to some epochs than others. This can be called the *problem of cosmological cohesion*, that is, to try to connect different cosmologies together to form a consistent frame for cosmic history, *a cohesive cosmology*, to compare with observations and other constraints. For example, suppose that an (FRW/GR/fluid) cosmology is valid after the Planck time onwards and that some (Bianchi/String/Vacuum) cosmology holds well before that time. The cosmological cohesion problem in this case is to connect the physically meaningful classical solutions of the two cosmology branches into one cohesive cosmology that would describe the entire cosmic history and be compatible with observations and other constraints. Theorems that would describe precisely what happens in such simple problems are lacking at present, but could greatly contribute to our understanding of the behaviour of candidate cosmologies.

The second offshoot of the range of validity issue stems from the fact that the solution spaces of different cosmologies are, in general, not isomorphic. For example, consider two cosmologies characterized by the same spacetime and matterfield structure but different gravity theories<sup>4</sup>. How are their solution spaces related? This question raises another one: How do we compare two cosmologies? The importance of the possible answers to such questions is obvious for we could transfer known results from one cosmology to another augmenting in this way our knowledge of the basic 'cosmology atlas'<sup>5</sup>.

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<sup>4</sup>It follows from the conformal equivalence theorem [23] that the solution space of, for instance, the family ( $\cdot$ /GR/scalarfield) is properly contained in the solution space of the family ( $\cdot$ /HDG/scalarfield) as the latter contains the ( $\cdot$ /HDR/vacuum) family (obtained by setting the scalarfield equal to zero) and this in turn is conformally equivalent to ( $\cdot$ /GR/scalarfield).

<sup>5</sup>The related issue of the conservation laws of two comparable cosmologies is also of interest as it is intimately connected to the problem of singling out of two conformally related cosmologies one which contains the true, physical metric that can be consistently used to measure times and distances (see, [24] for a recent review).

## 7 Mathematical cosmology in retrospect

*All science is cosmology, I believe. (Karl Popper)*

Understanding the dynamics of general classes of cosmologies,  $\mathcal{C} = (\mathbf{CS}, \mathbf{TG}, \mathbf{MF})$ , is the central problem of mathematical cosmology. In its most general form this problem is clearly intractable at present. There are three equivalent ways of rephrasing this problem splitting it into three components as follows:

1. The study of geometric and dynamical properties of a fixed class of spacetimes in the  $(\mathbf{TG}, \mathbf{MF})$ -space
2. The study of a fixed theory of gravity in the  $(\mathbf{CS}, \mathbf{MF})$ -space
3. The evolution properties of a fixed class of matterfields in the  $(\mathbf{CS}, \mathbf{TG})$ -space

Each of the three aspects above represents a different projection of the general problem of mathematical cosmology. As an exercise the reader is invited to describe the differences between the three components of the general cosmological problem!

Further reductions and simplifications in each of these components takes us too far afield and into other (most!) domains of pure or applied mathematics. For example, consider ‘switching off’ the  $\mathbf{TG}$  part in 3. Then one is left with an evolution equation for a ‘matterfield’ in a  $\mathbf{CS}$  (and if we further neglect the time coordinate we end up with a PDE for the matterfield in some specified Riemannian space). Also pure differential geometry can be thought of as the limit obtained from 1 when we switch off both  $\mathbf{TG}$  and  $\mathbf{MF}$  and leave only the ‘space part’ of the problem. Finally, any problem in the calculus of variations can be arrived at from 2 by suitable modifications. (Exercise!)

Theories of gravity, spacetimes and matterfields are the nuts and bolts of the mathematical cosmologist in his/her attempts to construct models that have the potential to consistently describe the universe, the cosmologies. Mathematical cosmology is a vast edifice in geometry, and dynamical systems theory plays a central role in all attempts to unravel the behaviour of every possible cosmology. Today mathematical cosmologists have a well-defined and consistent framework to exercise their imagination and special mathematical skills in their efforts to replace current puzzling issues awaiting for solution by new and more interesting ones. As Poincaré once put it: ‘mathematicians do not

destroy the obstacles with which their science is spiked, but simply push them toward its boundary’.

I wish to thank Peter Leach and John Miritzis for useful discussions and their critical readings of an initial version of this manuscript.

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